Exercise 16

Find the inverse Laplace transform of the following:

$$F(s) = \frac{1}{s(s^2+1)} - \frac{1}{s(s^2-1)}$$

Solution

Before taking the inverse Laplace transform of $F(s)$, factor $1/s$.

$$F(s) = \frac{1}{s}\left(\frac{1}{s^2+1} - \frac{1}{s^2-1}\right)$$

We will use the convolution theorem, which states

$$\mathcal{L}\left\{\int_0^x f_1(x-t)f_2(t)\,dt\right\} = F_1(s)F_2(s).$$

In this case, $F_1(s)$ is $1/s$ and $F_2(s)$ is the function in parentheses. Their inverse Laplace transforms are relatively easy to find.

$$F_1(s) = \frac{1}{s} \quad \rightarrow \quad \mathcal{L}^{-1}\{F_1(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1 = f_1(x)$$

$$F_2(s) = \frac{1}{s^2+1} - \frac{1}{s^2-1} \quad \rightarrow \quad \mathcal{L}^{-1}\{F_2(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2+1} - \frac{1}{s^2-1}\right\} = \sin x - \sinh x = f_2(x)$$

By the convolution theorem then,

$$\begin{aligned}
F(s) &= \mathcal{L}\left\{\int_0^x (\sin t - \sinh t)\,dt\right\} \\
&= \mathcal{L}\left\{\int_0^x \sin t\,dt - \int_0^x \sinh t\,dt\right\} \\
&= \mathcal{L}\left\{(-\cos t)\Big|_0^x - \cosh t\Big|_0^x\right\} \\
&= \mathcal{L}\{1 - \cos x - \cosh x + 1\} \\
&= \mathcal{L}\{2 - \cos x - \cosh x\}.
\end{aligned}$$

Finally, take the inverse Laplace transform of $F(s)$.

$$\begin{aligned}
\mathcal{L}^{-1}\{F(s)\} &= \mathcal{L}^{-1}\{\mathcal{L}\{2 - \cos x - \cosh x\}\} \\
&= 2 - \cos x - \cosh x
\end{aligned}$$